

# Nonperturbative solution of Yukawa theory and gauge theories\*

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**Abstract.** Recent progress in the nonperturbative solution of (3+1)-dimensional Yukawa theory and quantum electrodynamics and (1+1)-dimensional super Yang–Mills theory is summarized.

## 1 Introduction

Field theories can be solved nonperturbatively when quantized on the light cone [1, 2, 3]. This is done in a Hamiltonian formulation which, unlike Euclidean lattice gauge theory [4], yields wave functions. The properties of an eigenstate can then be computed relatively easily. Success came easily for two-dimensional theories [3], but in three or four dimensions the added difficulty of regularization and renormalization has until recently limited the success of the approach. Here we discuss recent progress with two different yet related approaches to regularization. One is the use of Pauli–Villars (PV) regularization [5] and the other, supersymmetry [6].

## 2 Pauli–Villars regularization

The most important aspect of the PV approach is the introduction of negative metric PV fields to the Lagrangian, with couplings only to null combinations of PV and physical fields. This choice eliminates instantaneous fermion terms from the Hamiltonian and, in the case of QED, permits the use of Feynman gauge without inversion of a covariant derivative. In addition, the Hamiltonian eigenvalue problem can be formulated in terms of transverse polar coordinates, which allow direct construction of eigenstates of  $J_z$  and explicit factorization from the wave function of the dependence on the polar angle; this reduces the effective space dimension and the size of the numerical calculation. The numerical approximation requires the introduction of special discretizations rather than the traditional momentum grid with equal spacings used in discrete light-cone quantization (DLCQ) [2, 3]. The special discretization allows the capture of rapidly

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varying integrands in the product of the Hamiltonian and the wave function, which occur for large PV masses.

An analysis of Yukawa theory with a PV scalar and a PV fermion, but without antifermion terms, is given in Refs. [7] and [8]. No instantaneous fermion terms appear in the light-cone Hamiltonian, because they are individually independent of the fermion mass and cancel between instantaneous physical and PV fermions. For a positive-helicity dressed fermion state  $\Phi_+$ , the wave functions satisfy the coupled system of equations that results from the Hamiltonian eigenvalue problem  $P^+ P^- \Phi_+ = M^2 \Phi_+$ . Each wave function has a total  $L_z$  eigenvalue of 0 (1) for a constituent-fermion helicity of  $+1/2$  ( $-1/2$ ).

Truncation to one boson leads to an analytically solvable problem [7]. The one-boson–one-fermion wave functions then have simple explicit forms, and the bare-fermion amplitudes must obey an algebraic equation which can be readily solved. An analysis of this solution is given in Ref. [7].

In a truncation to two bosons, we obtain reduced equations for the one-boson–one-fermion wave functions [8]. These reduced integral equations are converted to a matrix equation via quadrature and then diagonalized. The diagonalization yields the bare coupling as an eigenvalue and the discrete wave functions from the eigenvector. From the wave functions we can obtain any property of the state.

We apply these same techniques to QED in Feynman gauge [9]. In addition to the absence of instantaneous fermion interactions, we find that the constraint equation for the nondynamical fermion field is independent of the gauge field and can therefore be solved without inverting a covariant derivative. The resulting Hamiltonian is given in Ref. [9].

We consider the dressed electron state, without pair contributions and truncated to one photon. The one-photon–one-electron wave functions again have simple forms, and the bare-electron amplitudes satisfy algebraic equations nearly identical to those found in Yukawa theory. An analytic solution is again obtained. From this solution we can compute various quantities, including the anomalous magnetic moment [9].

### 3 Supersymmetric theories

For super Yang–Mills (SYM) theory, the technique used is supersymmetric discrete light-cone quantization (SDLCQ) [10, 6]. This method is applicable to theories with enough supersymmetry to be finite. The supersymmetry is maintained exactly within the numerical approximation by discretizing the supercharge  $Q^-$  and computing the discrete Hamiltonian  $P^-$  from the superalgebra anticommutator  $\{Q^-, Q^-\} = 2\sqrt{2}P^-$ . To limit the size of the numerical calculation, we work in the large- $N_c$  approximation; however, this is not a fundamental limitation of the method.

The stress-energy correlation function for  $\mathcal{N}=(8,8)$  SYM theory can be calculated on the string-theory side [11]:  $F(x^-, x^+) \equiv \langle T^{++}(x)T^{++}(0) \rangle = (N_c^{3/2}/g)x^{-5}$ . We find numerically that this is *almost* true in  $\mathcal{N}=(2,2)$  SYM theory [12]. By analyzing the Fourier transform with respect to the total momentum  $P^+ = K\pi/L$ , where  $K$  is the integer resolution and  $L$  the length scale

of DLCQ [2], we obtain [12]

$$F(x^-, x^+) = \sum_i \left| \frac{L}{\pi} \langle 0 | T^{++}(K) | i \rangle \right|^2 \left( \frac{x^+}{x^-} \right)^2 \frac{M_i^4}{8\pi^2 K^3} K_4(M_i \sqrt{2x^+ x^-}). \quad (1)$$

We then continue to Euclidean space by taking  $r = \sqrt{2x^+ x^-}$  to be real. The matrix element  $(L/\pi) \langle 0 | T^{++}(K) | i \rangle$  is independent of  $L$ . Its form can be substituted directly to give an explicit expression for the correlator.

The correlator behaves like  $(1 - 1/K)/r^4$  at small  $r$ . For arbitrary  $r$ , it can be obtained numerically by either computing the entire spectrum (for “small” matrices) or using Lanczos iterations (for large) [13]. We find [12] that for intermediate values of  $r$ , the correlator behaves like  $r^{-4.75}$ , or almost  $r^{-5}$ . The size of this intermediate region increases as  $K$  is increased.

We next consider  $\mathcal{N}=(1,1)$  SYM theory at finite temperature [14]. From the discrete form of the supercharge  $Q^-$  we can compute the spectrum, which at large- $N_c$  represents a collection of noninteracting modes. With a sum over these modes, we can construct the free energy at finite temperature from the partition function [15]  $e^{-p_0/T}$ . We obtain the total free energy as [14]

$$\mathcal{F}(T, V) = -\frac{(K-1)\pi}{4} VT^2 - \frac{2VT}{\pi} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} M_n \frac{K_1\left((2l+1)\frac{M_n}{T}\right)}{(2l+1)}. \quad (2)$$

The sum over  $l$  is well approximated by the first few terms. We can represent the sum over  $n$  as an integral over a density of states:  $\sum_n \rightarrow \int \rho(M) dM$  and approximate  $\rho$  by a continuous function. The integral over  $M$  can then be computed by standard numerical techniques. We obtain  $\rho$  by a fit to the computed spectrum of the theory and find  $\rho(M) \sim \exp(M/T_H)$ , with  $T_H \sim 0.845\sqrt{\pi/g^2 N_c}$ , the Hagedorn temperature [16]. From the free energy we can compute various other thermodynamic functions up to this temperature [14].

#### 4 Future work

Given the success obtained to date, these techniques are well worth continued exploration. In Yukawa theory, we plan to consider the two-fermion sector, in order to study true bound states. For QED the next step will be inclusion of two-photon states in the calculation of the anomalous moment. For SYM theories, we are now able to reach much higher resolutions; this will permit continued reexamination of theories where previous calculations were hampered by low resolution.

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